Class 3 – Photonic Crystals (October 8, 2010)

Assignment

In this assignment, we will develop a basic framework to describe the electromagnetic eigenfunctions in a one-dimensional photonic crystal. We will derive the formation of a photonic bandgap: a range of frequencies for which light can not propagate normal to the crystal planes.

a) Starting from Maxwell's equations in the absence of free charges and currents, derive the wave equation

$$\frac{1}{\varepsilon}\nabla\times\left(\nabla\times\mathbf{E}\right)=\left(\frac{\omega}{c}\right)^{2}\mathbf{E}\,. \qquad (1)$$

Equation (1) describes an eigenvalue problem for the electric field. The solution can be written as an expansion in plane waves through a Fourier transform:

$$\mathbf{E}(\mathbf{x})=\int d^{3}k\;\mathbf{A}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \qquad (2)$$

where **A(k)** is the amplitude of a plane wave with wave vector **k**.

In a photonic crystal, the dielectric constant is periodic;

$$\varepsilon(\mathbf{x})=\varepsilon(\mathbf{x}+\mathbf{a})\,, \qquad (3)$$

where **a** is an arbitrary lattice vector. For simplicity, in this assignment we will consider a one-dimensional photonic crystal (for example a stack of alternating layers of two different dielectrics), that is periodic in the *z* direction with period *a*. The reciprocal lattice vectors **G** of the crystal are defined as

$$\mathbf{G}=ng\hat{\mathbf{z}}=n\frac{2\pi}{a}\hat{\mathbf{z}},\quad n=0,\pm1,\pm2,... \qquad (4)$$

b) Show that

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{A}(\mathbf{k}))+\left(\frac{\omega}{c}\right)^{2}\sum_{\mathbf{G}}\beta_{\mathbf{G}}\mathbf{A}(\mathbf{k}-\mathbf{G})=0\,, \qquad (5)$$

where $\beta_{\mathbf{G}}$ are the Fourier coefficients of the dielectric function, $\varepsilon(\mathbf{x})$.

We now want to understand Bloch's theorem. Bloch's theorem states that in a periodic medium with period *a*, eigenfunctions of the electric field can be described by

$$\mathbf{E}(\mathbf{x})=\mathbf{E}_{\mathbf{K}}(\mathbf{x})e^{-i\mathbf{K}\cdot\mathbf{x}}, \qquad (6)$$

where $\mathbf{E}_{\mathbf{K}}$ is periodic; $\mathbf{E}_{\mathbf{K}}(\mathbf{x})=\mathbf{E}_{\mathbf{K}}(\mathbf{x}+\mathbf{a})$. It depends on the wavevector **K**, which is known as the Bloch wavevector.

- **c)** Consider eq. (5). The solutions to eq. (5) should be written in the form of a plane wave expansion. If the plane wave with a particular wavevector **K** appears in the eigensolution (2), explain which other plane waves are part of that solution. Using this information and the plane wave expansion of the field (eq. (2)), write $\mathbf{E}_\mathbf{K}(\mathbf{x})$ in terms of the coefficients **A**. Explain that you have now derived Bloch's theorem.

In the rest of this assignment we will assume wave propagation in the *z* direction. The photonic crystal is homogeneous in the *x* and *y* directions, and $\varepsilon(z)$ is always real and positive. To describe wave propagation in the crystal, it is necessary to find a dispersion relation that relates ω and K.

- **d)** Solving equation (5) can yield both the dispersion relation and the eigenfunctions $\mathbf{E}_\mathbf{K}$. Equation (5) is in principle an infinite set of coupled equations, but in practice only a few equations are needed to provide an approximate result.
  Expand the dielectric function only up to the first three terms; β_0, β_{-1}, and β_1 (where the subscript denotes n in eq. (4)). Note that $\mathbf{k} \cdot A = 0$. Near the Bragg condition, the dominant terms in the equations following from eq. (5) are only those with $A(K)$ and $A(K-g)$.
  Show that the dispersion relation near the Bragg condition is approximated by:

$$(\omega/c)^4(\beta_0{}^2 - \beta_1\beta_{-1}) - (\omega/c)^2\beta_0(K^2 + (K-g)^2) + K^2(K-g)^2 = 0 \qquad (7)$$

  Show that when the Bragg condition is satisfied exactly, two roots for ω^2 may be obtained:

$$\omega_\pm{}^2 = \frac{\pi^2 c^2}{a^2(\beta_0 \pm |\beta_1|)} \qquad (8)$$

The frequencies between ω_+ and ω_- are within the bandgap that has arisen from the coupling between the wave components $A(K)$ and $A(K-g)$.

- **e)** Give an expression for K for the center frequency of the stop gap (at $\omega = \pi c/(a\sqrt{\beta_0})$). Use an approximation assuming K is very close (but not equal) to $g/2$. Describe qualitatively what happens to a wave that is incident *from outside* onto a finite-sized photonic crystal at this frequency.

- **f)** The fractional bandgap (the frequency width of the stop gap divided by the center frequency in the stop gap) is a good measure of the 'photonic strength' of the photonic crystal, i.e., the coupling strength between $A(K)$ and $A(K\text{-}g)$ at the Bragg condition. Consider a one-dimensional photonic crystal that consists of alternating layers of equal thickness of two materials with refractive indices of 3.0 and 3.4. What is the magnitude of the fractional bandgap?

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